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Exact self-similar solutions for the two-dimensional plasma-ion sheath system

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Abstract

We present explicit self-similar solutions for the two-dimensional plasma-ion sheath system in the plasma physics. We show that the self-similar solutions are consistent with numerical simulations of Sheridan (1996 *Phys. Plasmas* **3** 2461–6).

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1. Introduction

The purpose of this paper is to present exact self-similar solutions for the motion of a plasma sheath (in short, sheath) in a plasma consisting of ions and electrons. The motion occurs in two space dimensions x, y and the self-similar solutions are functions of the self-similar independent variables $\xi = x/t$, $\eta = y/t$, where t is time. While considerable numerical and experimental evidence has appeared in [4, 5, 17, 19–22], there seem to be no earlier exact solutions in the literature. As is well known, sheaths arise from the breakdown of the quasi-neutrality, when boundary conditions are imposed on ion–electron plasmas [1, 8, 9, 12, 14–16]. Specifically sheath formation occurs when a large voltage drop occurs in a plasma, causing relatively light electrons to accelerate into a target boundary. The target then becomes strongly negatively charged, hence repelling other electrons and forming an electron vacuum boundary layer. Mathematically the breakdown may be seen via the study of steady state or dynamic equations. Although the analysis of sheath formation via the steady flow is quite well known [8, 12, 14], here we follow a dynamic model given in [16].

Consider a plasma consisting of ions and electrons. Then the density of the ions is denoted by n, the density of the electrons is $e^{-\Phi}$ (the Boltzmann relation for light electrons), $-\Phi$ is the electron potential and u is the velocity of the ions. Here all quantities have been made dimensionless (see e.g. [14–16]). In this case, a fluid description of an isothermal motion is provided by the isothermal Euler–Poisson equations

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \tau \nabla \ln n = \nabla \Phi, \\ \varepsilon^2 \Delta \Phi = n - e^{-\Phi}, \end{cases} \tag{1.1}$$

where ε is proportional to the Debye length and $\tau \geqslant 0$ is the ratio of ion temperature to electron temperature.

Since the Debye length is small, ε is a small parameter and the Poisson equation (1.1*c*) suggests the quasi-neutral relation $n = e^{-\Phi}$ should pervade in our problem. Substitution of this relation into (1.1*b*) yields the quasi-neutral system

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + (\tau + 1)\nabla \ln n = 0. \end{cases} \tag{1.2}$$

If (1.1) and (1.2) are supplemented by arbitrary initial and boundary conditions, the resulting initial-boundary value problem for (1.2) is generally ill-posed. Consider for example the cases of planar, cylindrically and spherically symmetric motions with boundary configurations consistent with these symmetries. In these cases, the quasi-neutral system (1.2) becomes

$$\begin{cases} \partial_t \rho + \partial_r(\rho u) = 0, \\ \partial_t u + \partial_r \left(\dfrac{u^2}{2} + (\tau + 1) \ln n \right) = \dfrac{\nu}{r}, \end{cases} \tag{1.3}$$

where $\nu = 0, 1, 2$ correspond to planar, cylindrical and spherical cases respectively and $\rho = r^\nu n$. System (1.3) possesses characteristic curves χ_1, χ_2 satisfying the relations

$$\frac{\mathrm{d}\chi_1}{\mathrm{d}t} = u - \sqrt{\tau + 1} \qquad \text{and} \qquad \frac{\mathrm{d}\chi_2}{\mathrm{d}t} = u + \sqrt{\tau + 1}. \tag{1.4}$$

Consider the case of planar motion ($\nu = 0$) with initial data (ρ, u, Φ) prescribed on $r > 0, t = 0$, with $u < -\sqrt{\tau + 1}$, initially and boundary data (u, Φ) prescribed on the target boundary $r = 0$, $t > 0$, $\Phi \gg 1$. In this case, both characteristics will run (at least for small $r > 0$, $t > 0$) from the initial line $r > 0$, $t = 0$ into the boundary $r = 0$, $t > 0$, thus transferring initial data into boundary data, and making the initial-boundary value problem overdetermined, and hence ill-posed. This ill-posedness reflects the breakdown of the quasi-neutral limit near the Bohm velocity $u = -\sqrt{\tau + 1}$ where the flow becomes supersonic (or more generally a Mach surface, see [23] and section 4 of this paper). Also note that since the Poisson equation (1.1*c*) reads

$$\varepsilon^2 \Delta \Phi = n - e^{-\Phi}, \tag{1.5}$$

the quasi-neutrality condition is violated when the left-hand side of (1.5) becomes non-negligible. Via the obvious scaling of (1.1*c*) Godyak and Sternberg [9] postulated that this occurs in a region when the electric field $\nabla\Phi$ is of the order of ε^{-1} (see also [15, 16]). Hence we have two possible different ways of locating the sheath '*interface*', one based on a sonic boundary and the other based on the magnitude of the electric field.

In order to capture the dynamics of the sheath boundary layer, we return to the original Euler–Poisson system (1.1) and we rescale space and time to fast variables $(\bar{\mathbf{x}}, \bar{t})$:

$$\bar{\mathbf{x}} = \frac{\mathbf{x}}{\varepsilon} \qquad \text{and} \qquad \bar{t} = \frac{t}{\varepsilon}$$

so that the rescaled Euler–Poisson system becomes

$$\begin{cases} \partial_{\bar{t}} n + \nabla_{\bar{\mathbf{x}}} \cdot (n\mathbf{u}) = 0, \\ \partial_{\bar{t}} \mathbf{u} + (\mathbf{u} \cdot \nabla_{\bar{\mathbf{x}}})\mathbf{u} + \tau \nabla_{\bar{\mathbf{x}}} \ln n = \nabla_{\bar{\mathbf{x}}} \Phi, \\ \Delta_{\bar{\mathbf{x}}} \Phi = n - e^{-\Phi}. \end{cases} \tag{1.6}$$

Note that under rescaling, non-homogeneous terms on the right-hand side of (1.6), for instance, ionization in (1.1*a*) and friction in (1.1*b*) will be of the order of ε and hence legitimately neglected in our theory. Furthermore, we once again assume that our boundary and initial conditions will be consistent with our rescaling. Also, as mentioned earlier in the sheath boundary layer, the electron density $e^{-\Phi}$ will be negligible and it is convenient to drop this term in (1.6*c*) so that we recover the classical sheath system

$$\begin{cases} \partial_{\bar{t}} n + \nabla_{\bar{\mathbf{x}}} \cdot (n\mathbf{u}) = 0, \\ \partial_{\bar{t}} \mathbf{u} + (\mathbf{u} \cdot \nabla_{\bar{\mathbf{x}}})\mathbf{u} + \tau \nabla_{\bar{\mathbf{x}}} \ln n = \nabla_{\bar{\mathbf{x}}} \Phi, \\ \Delta_{\bar{\mathbf{x}}} \Phi = n, \end{cases} \tag{1.7}$$

where $\tau \geqslant 0$ is a constant.

2. Derivation of the self-similar Euler–Poisson system

We introduce new dependent variables N and ϕ:

$$N := \bar{t}^2 n \qquad \text{and} \qquad \phi := \Phi - 2\ln \bar{t},$$

and assume that the velocity field $\mathbf{u} = (u, v)$ is irrotational so that there exists a potential function U for a vector field $(u - \xi, v - \eta)$:

$$\partial_\xi U = u - \xi \qquad \text{and} \qquad \partial_\eta U = v - \eta.$$

In the following, we derive the '*self-similar Euler–Poisson system*':

$$\begin{cases} \widetilde{\nabla} \cdot (N\widetilde{\nabla} U) = 0, \\ \phi = \dfrac{|\widetilde{\nabla} U|^2}{2} + U + \tau \ln N, \\ \widetilde{\Delta} \phi = N - e^{-\phi}, \end{cases} \tag{2.1}$$

where $\widetilde{\nabla}$ and $\widetilde{\Delta}$ are taken with respect to self-similar variables (ξ, η).

- *Derivation of* (2.1*a*). By direct calculation, we have

$$\begin{aligned} \bar{t}^3(\partial_t n + \nabla \cdot (n\mathbf{u})) &= -\xi \partial_\xi N - \eta \partial_\eta N - 2N + \partial_\xi (Nu) + \partial_\eta (Nv) \\ &= \partial_\xi (N(u - \xi)) + \partial_\eta (N(v - \eta)) \\ &= \partial_\xi (N \partial_\xi U) + \partial_\eta (N \partial_\eta U) \\ &= \widetilde{\nabla} \cdot (N \widetilde{\nabla} U). \end{aligned}$$

  Hence (1.3*a*) implies

$$\widetilde{\nabla} \cdot (N\widetilde{\nabla} U) = 0.$$

- *Derivation of* (2.1*b*). Recall from (1.3) that the momentum equations for u and v:

$$\partial_{\bar{t}} u + u\partial_{\bar{x}} u + v\partial_{\bar{y}} u + \tau \partial_{\bar{x}} (\ln n) = \partial_{\bar{x}} \phi, \tag{2.2}$$

$$\partial_{\bar{t}} v + u\partial_{\bar{x}} v + v\partial_{\bar{y}} v + \tau \partial_{\bar{y}} (\ln n) = \partial_{\bar{y}} \phi, \tag{2.3}$$

  where we used $\nabla_{\bar{\mathbf{x}}} \phi = \nabla_{\bar{\mathbf{x}}} \Phi$. By direct calculation, (2.2) reduces to

$$\partial_\xi \left(\frac{(\partial_\xi U)^2}{2} + U + \frac{(\partial_\eta U)^2}{2} + \tau \ln N - \phi \right) = 0. \tag{2.4}$$

  Similarly, we have

$$\partial_\eta \left(\frac{(\partial_\eta U)^2}{2} + U + \frac{(\partial_\xi U)^2}{2} + \tau \ln N - \phi \right) = 0. \tag{2.5}$$

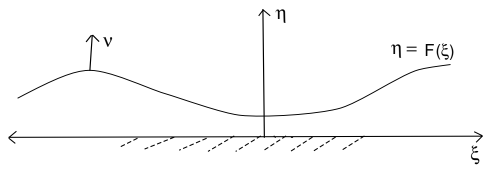

Figure 1. A schematic diagram of an infinite plane target boundary.

We combine (2.4) and (2.5) to obtain

$$\phi = \frac{|\widetilde{\nabla} U|^2}{2} + U + \tau \ln N,$$

where of course ϕ is determined only up to a constant.

- *Derivation of* (2.1*c*). The Poisson equation becomes

$$\Delta_{\bar{\mathbf{x}}}\phi = \partial_{\bar{x}}^2\phi + \partial_{\bar{y}}^2\phi = \frac{\partial_\xi^2\phi + \partial_\eta^2\phi}{\bar{t}^2} = \frac{\widetilde{\Delta}\phi}{\bar{t}^2}.$$

Hence we have

$$\widetilde{\Delta}\phi = \bar{t}^2\Delta\phi = \bar{t}^2(n - e^{-\Phi}) = N - e^{-\phi}.$$

Now let us consider our quasi-neutral system, i.e., (2.1) where $N = e^{-\phi}$ (or $n = e^{-\Phi}$):

$$\begin{cases} \widetilde{\nabla} \cdot (N\widetilde{\nabla} U) = 0, \\ \dfrac{|\widetilde{\nabla} U|^2}{2} + U + (\tau + 1)\ln N = 0, \end{cases} \tag{2.6}$$

where again U is determined only up to a constant. If we substitute (2.6*b*) into (2.6*a*), we obtain the second-order equation

$$[\tau + 1 - (\partial_\xi U)^2]\partial_\xi^2 U + [\tau + 1 - (\partial_\eta U)^2]\partial_\eta^2 U - 2(\partial_\xi U)(\partial_\eta U)(\partial_\xi\partial_\eta U) = (\partial_\xi U)^2 + (\partial_\eta U)^2. \tag{2.7}$$

System (2.7) is elliptic when $|\widetilde{\nabla} U| < \sqrt{\tau + 1}$ and hyperbolic when $|\widetilde{\nabla} U| > \sqrt{\tau + 1}$.

3. Explicit self-similar sheath solutions

In this section we consider explicit solutions to the self-similar sheath system:

$$\begin{cases} \widetilde{\nabla} \cdot (N\widetilde{\nabla} U) = 0, \\ \phi = \dfrac{|\widetilde{\nabla} U|^2}{2} + U + \tau \ln N, \\ \widetilde{\Delta}\phi = N, \end{cases} \tag{3.1}$$

where we have assumed that $e^{-\phi}$ is negligible in the Euler–Poisson system (2.1). For example, the target boundary may be given by

$$\partial\Omega_0 = \{(\xi, \eta) : \xi \in \mathbb{R},\ \eta = 0\},$$

which is an infinite plane (see figure 1) or a wedge $\partial\Omega_0 = \{(\xi, \eta) : \xi \leqslant 0, \eta = 0; \eta = (\tan\theta_w)\xi, \xi \geqslant 0\}$.

Below we look for (N, U) with the following ansatz

$$N = N_0 \qquad \text{and} \qquad U = A\xi\eta + \frac{B}{2}(\xi^2 - \eta^2) + C\xi + D\eta + E, \tag{3.2}$$

where N_0, A, B, C, D and E are constants. Since N is a constant and by the ansatz for U,

$$\widetilde{\nabla} \cdot (N\widetilde{\nabla}U) = 0.$$

We substitute ansatz (3.2) to the the third equation of the self-similar sheath system to find N_0:

$$\begin{aligned}
\widetilde{\Delta}\phi &= \widetilde{\Delta}\left(U + \frac{|\widetilde{\nabla}U|^2}{2} + \tau \ln N_0\right) \\
&= \widetilde{\Delta}\left(\frac{|\widetilde{\nabla}U|^2}{2}\right) \\
&= \frac{1}{2}\widetilde{\Delta}((A\eta + B\xi + C)^2 + (A\xi - B\eta + D)^2) \\
&= 2(A^2 + B^2) = N_0.
\end{aligned} \tag{3.3}$$

Hence we have determined self-similar solutions for the system of the form

$$\begin{aligned}
&N = 2(A^2 + B^2), \qquad U = A\xi\eta + \frac{B}{2}(\xi^2 - \eta^2) + C\xi + D\eta + E, \\
&\phi = A\xi\eta + \frac{B}{2}(\xi^2 - \eta^2) + C\xi + D\eta + E + \frac{1}{2}((A\eta + B\xi + C)^2 \\
&\qquad + (A\xi - B\eta + D)^2) + \tau \ln 2(A^2 + B^2).
\end{aligned} \tag{3.4}$$

4. Location of the sheath interface

In this section, we look for the explicit self-similar profile F using the interface condition.

Recall from our discussion in section 1 that earlier work has suggested the choice of the sheath edge as either (i) a Mach surface or (ii) the region where quasi-neutrality begins to break down. However to use (ii) would mean solving the classical isothermal Euler equations (1.2) under the scaling $\xi = x/t$, $\eta = y/t$. Furthermore, relating this solution to any boundary conditions would add an additional complication. Hence for simplicity we choose to use (i) the Bohm-type criterion and identify the sheath interface as the curve in the self-similar (ξ, η) domain where the normal component of the pseudo-velocity $(\partial_\xi U, \partial_\eta U)$ is sonic. This is precisely the Mach surface definition formulated by Stangeby and Allen [23], and extended by us to the self-similar case [7]. We will assume that $\mathcal{S}$ may be represented by the graph

$$y = tF\left(\frac{\mathbf{x}}{t}\right)$$

so that

$$\eta = F(\xi) \qquad \text{and} \qquad \boldsymbol{\nu} = \frac{(-F', 1)}{\sqrt{1 + (F')^2}}, \tag{4.1}$$

where $\boldsymbol{\nu}$ is the normal to the sheath interface. Since

$$U = A\xi\eta + \frac{B}{2}(\xi^2 - \eta^2) + C\xi + D\eta + E,$$

we have

$$\partial_\xi U = A\eta + B\xi + C, \qquad \partial_\eta U = A\xi - B\eta + D.$$

On the other hand, the Bohm-type relation $(\partial_\xi U, \eta_\xi U) \cdot \boldsymbol{\nu} = -\sqrt{1 + \tau}$ implies

$$(AF + B\xi + C, A\xi - BF + D) \cdot (-F', 1) = -\sqrt{(1 + \tau)(1 + (F')^2)}. \tag{4.2}$$

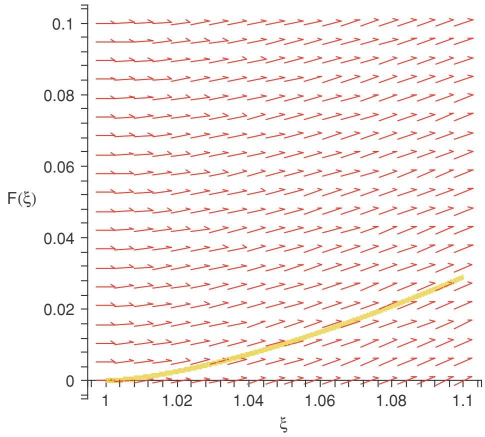

Figure 2. One branch of the sheath edge corresponding to (4.4).
(This figure is in colour only in the electronic version)

We set

$$\mathcal{A} := (1+\tau) - (AF + B\xi + C)^2,$$
$$\mathcal{B} := 2(AF + B\xi + C)(A\xi - BF + D),$$
$$\mathcal{C} := (1+\tau) - (A\xi - BF + D)^2.$$

We see that (4.2) yields a first order ODE for F:

$$F' = \frac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}}. \tag{4.3}$$

Note that knowledge of F, U and ϕ provides us *a posteriori* with self-similar solutions for boundary value problems. For example consider the case $A = 1$, $B = C = D = E = 0$, so that in the plasma sheath (via (3.4))

$$N = 2, \qquad U = \xi\eta, \qquad \phi = \xi\eta + \tfrac{1}{2}(\xi^2 + \eta^2) + \tau \ln 2.$$

On a '*wall*' $\eta = 0$, we have

$$\phi = \frac{\xi^2}{2} + \tau \ln 2, \qquad U = 0, \qquad \partial_\xi U = 0, \qquad \partial_\eta U = \xi$$

and the flow is

$$\text{supersonic when } |\xi| > \sqrt{\tau+1} \qquad \text{and subsonic when } |\xi| < \sqrt{\tau+1}.$$

Equation (4.3) for the sheath edge becomes

$$F' = \frac{-F\xi + \sqrt{1+\tau}\sqrt{F^2 + \xi^2 - (1+\tau)}}{(1+\tau) - F^2}, \tag{4.4}$$

which leads to a sheath edge for $|\xi| > \sqrt{\tau+1}$. A numerical integration for (4.4) with initial conditions

$$F(\sqrt{1+\tau}) = 0 \qquad \text{and} \qquad F(-\sqrt{1+\tau}) = 0$$

yields the two branches of the sheath edge (see figure 2, $F(1) = 0$ with $\tau = 0$).

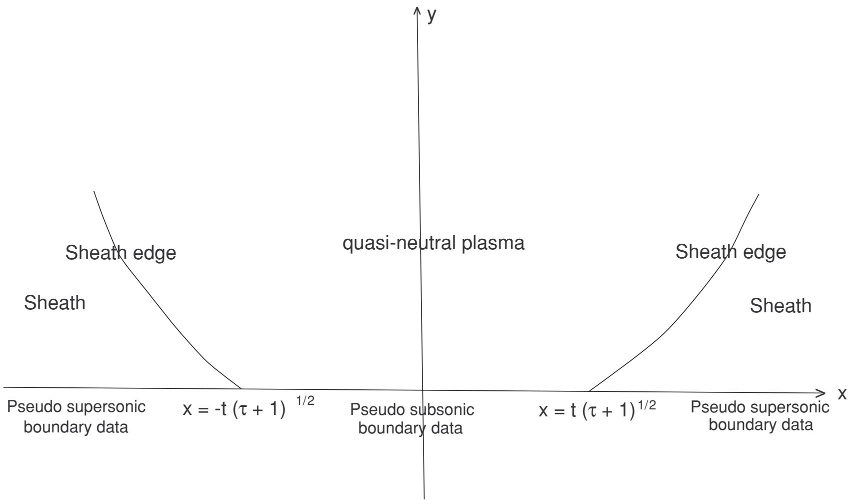

Figure 3. Hypothetical experimental illustration of the solution in section 4

5. Conclusion

In this paper, we have rewritten quasi-neutral and sheath dynamics in terms of self-similar independent variables $\xi = x/t$, $\eta = y/t$ and dependent variables N, U:

$$\partial_\xi U = u - \xi, \qquad \partial_\eta U = v - \eta, \qquad N = \bar{t}^2 n, \quad \phi = \Phi - 2\ln \bar{t}.$$

Furthermore, our formulation has led to a self-similar step sheath model (in fact a self-similar version of the one of Riemann and Daube [16]) whose transition sheath edge is on the Mach-edge curve $\widetilde{\nabla} U \cdot \nu = -\sqrt{\tau + 1}$, where τ is the ratio of ion to electron temperatures. An exact solution in the sheath layer has been given:

$$U = A\xi\eta + \frac{B}{2}(\xi^2 - \eta^2) + C\xi + D\eta + E, \qquad N = 2(A^2 + B^2)$$

and graph of the sheath edge $\eta = F(\xi)$ has been explicitly given for the case $A = 1$, $B = C = D = E = 0$.

Finally, we note the connection between our work and the motivating numerical results of Sheridan [20]. Sheridan's numerical results noted that self-similarity appears as a time-asymptotic result of his solution of the non-self-similar Euler–Poisson system (1.1) for a hypothetical plasma flowing into a target wall. Here we show that self-similarity is no surprise: the Euler–Poisson equations themselves admit a self-similar representation. As to a hypothetical experiment, ours would be slightly different from Sheridan's and follows from the data of section 4. This is illustrated in figure 3.

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